Irregular fixation – II. The orbits of irregular satellites

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ABSTRACT

Irregular satellites (ISs) are believed to have been captured during the Solar system’s dynamical history and provide clues for the Solar system’s formation and evolution. ISs occupy a large fraction of the Hill sphere of their host planet and their orbits are highly perturbed by the Sun. We use a novel formalism developed in Paper I to characterize their orbits in terms of an effective secular Hamiltonian (the Brown Hamiltonian) that accounts for their large orbital separations. We find that prograde satellites generally follow the Brown Hamiltonian, while retrograde satellites (which extend further) deviate more significantly. Nevertheless, the phase portrait is much better described by the Brown Hamiltonian for all satellites. We construct a semi-analytical criterion that predicts the librating orbit based on the effective energy due to the Brown Hamiltonian. We also check our results with highly accurate N-body integrations of satellite orbits, where initial conditions are loaded directly from the updated ephemeris from the NASA Horizons data base. Although the retrograde librating orbits occupy more area in the parameter space, the vast majority of librating ISs are prograde. Using our method, we find 13 librating satellites, 8 of them previously known to librate, and the rest shown to librate for the first time. Further observations of existing and new satellites could shed more light on the dynamical history of the Solar system and satellite formation and test our results.

Key words: chaos – celestial mechanics – planets and satellites: dynamical evolution and stability – stars: kinematics and dynamics.

1 INTRODUCTION

Irregular satellites (ISs) are small objects that are weakly bound to their planet hosts. The main source of perturbations of their orbits are the Sun and other planets, rather than the inner quadrupolar distortion of the planets’ oblateness and inner satellite system (Nesvorný et al. 2003). ISs usually have large eccentricities and inclinations, which vary significantly over time. Studying the orbits of ISs is vital for understanding the origin and dynamical evolution of the giant planets and the Solar system as a whole (Carruba et al. 2002).

The vast majority of IS orbits are retrograde (with the mutual inclination \( \iota \) between the two orbital planes being \( >90 \) deg), which also exceed further than prograde orbits (where \( \iota < 90 \) deg). The extended stability limit for retrograde orbits is due to the Coriolis force that counteracts the tidal shear from the Sun (Innanen 1979, 1980). This dichotomy is even more striking due to the lack of near-polar orbits near \( \iota \sim 90 \) deg (e.g. Carruba et al. 2002; Nesvorný et al. 2003).

Due to their large orbits, the dynamics of ISs have been studied in terms of the long-term evolution of three bodies (Hamilton & Burns 1991; Carruba et al. 2002). Čuk & Burns (2004) developed an effective secular theory that accounts for the relatively large period ratio between the satellite and the planet’s orbit. Frouard, Vienne & Fouchard (2011) studied the chaotic evolution of ISs in the context of secular and mean motion resonances between the different giant planets. The stability zone for arbitrary mutual inclinations is smallest for polar orbits (e.g. Grishin et al. 2017; Tory, Grishin & Mandel 2022). The early instability of polar orbits arises due to the von Zeipel–Lidov–Kozai (ZLK) effect (von Zeipel 1910; Kozai 1962; Lidov 1962), where periodic oscillations coherently change the eccentricity and inclination of the inner orbit on longer, secular time-scales.

In addition to destabilizing polar orbits, the ZLK effect is caused due to a fixed point in the eccentricity \( e_1 \) – argument of periapsis \( \omega_1 \) phase space, and an orbit can potentially librate around it with limited range for \( \omega_1 \), which is different from apsidal advance where \( \omega_1 \) completes full revolutions. With hundreds of ISs found to date, only a handful are found to be librating: Carpo and Euporie around Jupiter (Brozović & Jacobson 2017; Sheppard et al. 2023), Kiviuq, Ijiraq, and S2004, S31 around Saturn (Jacobson et al. 2022), Margaret around Uranus (Brozović & Jacobson 2009; Brozović & Jacobson 2022), and Sao and Neso around Neptune (Brozović, Jacobson & Sheppard 2011). Moreover, Carruba et al. (2002) identified hypothetical librating families at higher inclinations.

ISs form a sub-class of hierarchical triple systems, where the inner binary of semimajor axis \( a_1 \) is perturbed by a distant tertiary at semimajor axis \( a_2 \gg a_1 \). For a small period ratio \( P_1/P_2 \ll 1 \), an integrable, analytical, approximate solution for the ZLK effect can be found (Kinoshita & Nakai 2007). The latter approximation, however, is inaccurate for orbits with non-negligible period ratio, where the system is still stable, but the time-scale hierarchy is mild (Luo, Katz

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Expansions of the disturbing function of systems of ‘mild’ hierarchy have a long history, from the Lunar theory (Brown 1936a, c; Brouwer & Clemence 1961) to multiple stellar (Brown 1936b; Soderhjelm 1975) and compact object (Luo et al. 2016; Will 2021) systems. Tremaine (2023) had recently shown that the aforementioned (and other related) derivations are equivalent to the first instance of the ‘Brown Hamiltonian’, due to Brown (1936a). Generalized analogues of the Brown Hamiltonian for octupole and higher order potentials are given by Lei, Circi & Ortoire (2018).

For even larger period ratios, even the corrected double-averaged theory is insufficient, and additional higher order corrections are required. Beaugé, Nesvorný & Dones (2006) used a canonical perturbation theory, while Lei (2019) used a literal expansion of the disturbing function. In both cases, the averaging procedure is done after the canonical transformation of the osculating elements. This is contrary to the Brown Hamiltonian formalism, where first a single averaging over the inner orbit is applied before the canonical transformation (Luo et al. 2016; Tremaine 2023), which gives more accurate results where the difference between the outer and inner periods are blurry. While the Brown Hamiltonian is valid for any eccentricity, the higher order expansion relies on Legendre expansion eccentricities in terms of Hansen coefficients, which does not converge for eccentricity above $e_c = 0.66$ (see end of section 2 of Beaugé et al. 2006, and also Wintner 1941; Lei 2019).

In the first paper (Grishin 2024, hereafter Paper I), we derived an analytical expression for fixed points for the Brown Hamiltonian, which depend on dimensionless quantities constructed from the triple system’s architecture. We also explored the available parameter space for librating orbits if they are initialized at the fixed point. The energy associated with the Brown Hamiltonian is conserved on average, while the fluctuating terms are also proportional to the period ratio.

In this second paper in the series, we apply our results to explore the orbits of ISs. We formulate an effective energy condition for the orbits to be allowed to librate. We also confirm that the known librating ISs satisfy this condition and suggest that several other satellites are librating.

The paper is organized as follows: We first briefly review the main results of Paper I in Section 2. We describe how to select IS satellites from the NASA Jet Propulsion Laboratory (JPL) data in Section 3. In Section 4, we apply our result to ISs of the giant planets. We first devise a condition for orbits to be potentially librating in Section 4.1, and refine it in Section 4.2. The condition is based on the conservation of the effective energy due to the Brown Hamiltonian and on the amplitude of the osculating fluctuations of the orbital elements. In Section 5, we test it with direct $N$-body integrations and compare the orbital evolution of various ISs. Using this condition, we find all eight known librating satellites and suggest that five others are also librating. Finally, in Section 6, we conclude our results and discuss our limitations and potential avenues for future research directions.

## 2 Fixed Points and Librating Zones

Here, we briefly summarize the main results of Paper I.

Consider a hierarchical system with masses $m_0, m_1 \ll m_2$, orbital separations $a_1 \ll a_2$, and other orbital elements denoted by subscript 1 for the inner orbit and 2 for the outer orbit. In the quadrupole approximation, the quantity $j \equiv \sqrt{1 - e^2} \cos i$ is conserved. The motion of the satellite can be described by two terms. The first is the secular quadrupole Hamiltonian

$$\mathcal{H}_{sec} = \langle \mathcal{H}_1 \rangle = C \left[ 1 - 6e_1^2 + 3j_1^2 + 15e_1^2 \sin^2 i \sin^2 \omega_1 \right],$$

where $C = \frac{G \mu m_1}{8a_2 (1 - e_2^2)} \left( \frac{a_1}{a_2} \right)^2$.

The second term is the Brown Hamiltonian

$$\mathcal{H}_B = -\varepsilon_{SA} C \sum_{i=2}^8 \left[ \frac{(1 - j_i^2)}{3} + 8e_i^2 \sin^2 \omega_i \sin^2 \epsilon_i + O(e_i^4) \right].$$

Here,

$$\varepsilon_{SA} = \frac{a_1}{a_2} \frac{\mu m_1}{m_0 + m_1}.$$}

The associated inclination at the fixed point is

$$\cos \epsilon_{fix} = \frac{j_2}{a_{1/2}} = \text{sign}(j_2) \left( \frac{3}{5} \right)^{1/4} \sqrt{\frac{1 - \frac{8\varepsilon_{SA} j_2}{3}}{1 + \frac{8\varepsilon_{SA} j_2}{3}}} |j_2|^{1/2}.$$

Note that for $\varepsilon_{SA} = 0$ we retain the fixed point previously found for the secular ZLK solution (only for $\mathcal{H}_{sec}$; Antognini 2015; Hamers 2021).

A direct $N$-body integration shows that the energy given by $\mathcal{H}_{sec}$ is conserved on average, and fluctuates about a mean value with an amplitude proportional to $\varepsilon_{SA}$. Using a large grid of initial conditions, we initialized $\sim 3 \times 10^6$ three-body systems at the fixed points according to equation (4) (if they exist) in Paper I. We then integrated the systems and classified whether they were able to stay in libration. Besides the fractal boundary near the instability, we found a numerical fit for the boundary between libration and circulation in $j_2-\varepsilon_{SA}$ space:

$$\varepsilon_{SA}(j_2) = \left\{ \begin{array}{ll}
0.4(\sqrt{3/5} - j_2)^{0.7} & j_2 > 0; \varepsilon_{SA} \in [0.01, 0.16], \\
0.08 + 0.5(0.8 - j_2)^{0.52} & j_2 < 0; \varepsilon_{SA} \in [0.08, 0.23], \\
0.08 - 0.5(0.8 + j_2)^{0.52} & j_2 < 0; \varepsilon_{SA} \in [0.01, 0.08].
\end{array} \right.$$
osculating orbital elements. The mean elements are descriptive and cannot be used as initial conditions for N-body integrations (Broživić, private communication).

In order to load the initial conditions for the satellites, we use a built-in pipeline (https://rebound.readthedocs.io/en/latest/ipython_examples/Horizons) that directly reads and initializes an N-body system with the most up-to-date state vector provided by Horizons. The pipeline is publicly available as part of the REBOUND code (https://rebound.readthedocs.io/en/latest/), which is a highly accurate N-body code (Rein & Liu 2012). All the integrations in Section 5 were computed using the TASS5 adaptive time-step integrator, accurate to machine precision for billions of orbits (Rein & Spiegel 2014) and is built-in within REBOUND.

After setting up three-body systems, we calculate the orbits and use the calculated (osculating) orbital elements. We filter only satellites of the giant planets, which excludes the Moon, Mars’s two satellites, and Pluto’s five satellites. We then take only ISSs as follows: For each giant planet, we first calculate its Laplace radius (Tremaine, Touma & Namouni 2009)

\[
\rho_L = \left( \frac{j'_0 R_0^2 a_0^2 (1 - e_0^2)^{3/2} \frac{m_p}{M_0}} {} \right)^{1/5},
\]

where \(j'_0\) is the quadrupolar distortion coefficient, and \(R_0, a_0, e_0,\) and \(m_p\) are the giant planets’ radius, semimajor axis, eccentricity, and mass, respectively. We will use the values for \(j'_0\) that includes the contribution from the inner satellite system, found in table 1 of Tremaine et al. (2009). Satellites with orbits \(r < \rho_L\) will be subjected to the inner quadrupole, which causes apsidal advance and prevents ZLK oscillations (Liu, Muñoz & Lai 2015; Grishin, Lai & Perets 2018a; Grishin et al. 2020).

We deem the satellite irregular if \(a_1 > \rho_L\), which leaves us with 228 objects. This ensures that the dynamics are governed by the outer quadrupolar perturbations from the Sun, and not the inner quadrupolar distortion.

4 ENERGY CONDITIONS

4.1 Instantaneous energy condition

For each satellite, we calculate \(j_z\) and \(\epsilon_{SA}\) (from the osculating elements), and also the current energy:

\[
E_c = \mathcal{H}_{sec}(j_z, \epsilon_1, \omega_1) + \mathcal{H}_0(\epsilon_{SA}, j_z, \epsilon_1, \omega_1).
\]

We note that the osculating \(j_z\) and \(\epsilon_{SA}\) are different from their averaged values, \(\bar{j}_z\) and \(\bar{\epsilon}_{SA}\), respectively. Although \(j_z\) and \(\epsilon_{SA}\) are oscillating, we will treat them as constant for now. This first step is done to filter out orbits that cannot be librating, from ‘suspected’ orbits.

The total Hamiltonian essentially has one degree of freedom for the conjugate variables \(\epsilon_1^1\) and \(\omega_1\). The mutual inclination \(i\) is eliminated via \(\cos i = j_z/\sqrt{1 - e_z^2}\).

We also calculate the critical energy for the separatrix – the critical constant energy curve that separates circulating and librating orbits, which is evaluated at the origin:

\[
E_s = \mathcal{H}_{sec}(j_z, 0, 0) + \mathcal{H}_0(\epsilon_{SA}, j_z, 0, 0).
\]

In practice, \(\epsilon_1 = 0\) is an unstable hyperbolic fixed point, so we take \(\epsilon_1 = 10^{-5}\) and \(\omega_1 = 0\) to estimate \(E_s\) numerically. An orbit is suspected as librating if \(E_s > E_c\). Applying this procedure results in 33 satellites suspected as librating. It is important to stress that this is not a sufficient condition, since the osculating elements vary over time and the energy contours generated from the osculating elements are inaccurate.

We integrate all the IS orbits to check whether or not the criterion predicts a ZLK librating state. For simplicity, we set up three-body systems of the Sun, the planet, and the test particle, which is justified if the ISs are beyond the Laplace radius. We show in Section 5 that the known librating satellites have a similar orbital evolution to what had been reported in the past; thus, we believe that our approximate integrations capture the essence of the librating satellites.

In Fig. 1, we show the result of our analysis for the prograde case (right) and the retrograde case (left). We find that 13 of the 33 suspected satellites are librating. All prograde suspected satellites are in the allowed region for libration, while many of the retrograde suspected satellites are outside the allowed region. Note that there is no restriction where satellites can circulate in \(\epsilon_{SA} = j_z\) space (it depends on \(\omega_1\)), but libration is allowed only inside the green area. 10 of the librating satellites are prograde, while only 3 are retrograde. On the other hand, only 2 of the suspected prograde satellites are circulating, and the other 18 suspected and circulating orbits are retrograde.

We stress that the red lines are an empirical boundary that is obtained in Paper I, and does not come from the Brown Hamiltonian (although see the appendix in Paper I for a comparison to boundaries obtained from the Brown Hamiltonian). We see that besides Euporie, all the librating satellites are within the boundary, and many of the prograde ones lie close to it. Librating orbits move on stable islands and exhibit lower eccentricity variations; hence, they are expected to be more stable, which is consistent with our findings.

The main reason is that the retrograde orbits extend to much further distances (and larger \(\epsilon_{SA}\)), and additional perturbations cause the energy condition to be less accurate. High-order theory could in principle remedy the discrepancy. In particular, Lei (2019)’s model incorporates high-order secular theory, which has good correspondence. However, Lei (2019)’s model is hard to realize in practice due to its inherent complexity. Moreover, the literal expression of the separation in terms of mean anomalies and Hansen coefficients does not converge for eccentricity above \(e_z = 0.6627\) (Wintern 1941). While Neso’s eccentricities frequently exceed this value, S2021_N1’s eccentricity is marginal and Euporie’s eccentricity is relatively far from this threshold. It is possible in principle to construct a secular theory somewhat between the Brown Hamiltonian’s simplicity and Lei (2019)’s accuracy, but it is beyond the scope of this paper.

Inspecting the error bars in Fig. 1, we observe that the fluctuation in \(j_z\) is much larger than in \(\epsilon_{SA}\). A typical estimate of the fluctuations is \(2\Delta j_z\), where \(\Delta j_z\) is given by equation (13), and is around \(\sim 0.05\) \(\epsilon_{SA} e_z^2/4\). The proportionality constant is between 0.7 and 0.9 and depends on the remaining orbital elements. The largest variation of \(j_z\) is associated with the most eccentric satellites such as Neso and S20201_N1. For prograde satellites, Margaret and Carpo have also similar variation in \(j_z\), since the ratio square of the maximal (or average) eccentricities (0.9/0.65)\(^2\) \(\sim 1.9\) is comparable to the ratio of \(\epsilon_{SA} \sim 1.95\).

The latter diversity requires an additional examination and refinement of our condition, which is explored below.
4.2 Mean energy condition

To better quantify the possible orbits, we construct the following analysis: For each satellite, we calculate the dimensionless energy

$$\xi_c = \frac{E_c - E_s}{E_{\text{fix}} - E_s}. \quad (10)$$

Here,

$$E_{\text{fix}} = H_{\text{sec}}(j_z, e_{\text{fix}}, \pi/2) + H_{\theta}(\epsilon_{\text{SA}}, j_z, e_{\text{fix}}, \pi/2) \quad (11)$$

is the energy at the associated fixed point, and it is the maximal effective energy the satellite can have in its current orbit.

For orbits that are suspected as librating, \(\xi_c > 0\). The normalization by \(E_{\text{fix}} - E_s\) also guarantees that \(\xi_c \leq 1\), where equality is achieved only at the fixed point. In order to account for potential fluctuations, we estimate the variation in the energy via the fluctuation in \(j_z\). We denote

$$E_c^{\pm} = H_{\text{sec}}(j_z \pm \Delta j_z, e_1, e_{\text{SA}}) + H_{\theta}(\epsilon_{\text{SA}}, j_z \pm \Delta j_z, e_1, e_{\text{SA}}), \quad (12)$$

where the fluctuation \(\Delta j_z\) is given by equation (35) of Luo et al. (2016),

$$\Delta j_z = \frac{3\epsilon_{\text{SA}}}{8} \left[ 5e_1^2 \cos^2 \omega_1 + \frac{j_z^2}{1 - e_1^2} \sin^2 \omega_1 \right] + 1 - e_1^2 - j_z^2. \quad (13)$$

Note that in the limit of high eccentricity (and \(\sin \omega_1 = 1\)) we get back to equation (25) of Grishin et al. (2018b):

$$\lim_{e_1 \to 1, j_z \to 0} \Delta j_z = \frac{15}{8} \epsilon_{\text{SA}} e_1^2 \cos^2 \omega_1 = \frac{9}{8} \epsilon_{\text{SA}} e_1^2. \quad (14)$$

From \(E_c^{\pm}\), we can construct the relative energy fluctuation

$$\xi_c^{\pm} = \frac{E_c^{\pm} - E_s}{E_{\text{fix}} - E_s}. \quad (15)$$

Without loss of generality, we rearrange \(\xi_c^{\pm}\) such that \(\xi_c^{+} > \xi_c^{-}\).

Fig. 2 shows the dimensionless energy \(\xi_c\) (equation 10) for each of the suspected satellites together with the boundaries of the fluctuating energy given by \(\xi_c^{\pm}\). The condition for being potentially in libration is \(\xi_c \in [0, 1]\), where the extreme values are attained only if the initial energy is exactly at the separatrix (for 0–dashed red line) or at the fixed point (for 1–dashed green line). 6 out of the 13 librating satellites have always positive energy. Most of the circulating orbits have a large extent of \(\xi_c\) and can spend a significant amount of time below 0.

The boundary of \(\xi_c^{\pm}\) can extend beyond unity because the fluctuation in \(j_z\) shifts the Hamiltonian to a different system, and the new \(E_c^{\pm}\) can exceed the original \(E_{\text{fix}}\). Of course, \(E_c^{+}\) cannot exceed \(E_{\text{fix}}\), the energy at the fixed point evaluated for at \(j_z = \Delta j_z/2\).

Fig. 3 shows the same data in \(\xi_c^{-}\)–\(\Delta \xi\) space where

$$\Delta \xi = \left| \frac{\xi_c^{-}}{\xi_c^{\pm}} \right| - \left| \frac{E_c^{-} - E_s}{E_c^{\pm} - E_s} \right| \quad (16)$$

is the relative extent of the fluctuation. We discard satellites with \(\xi_c^{-} > 0\) since they are always librating. The former measures how deep the fluctuating energy can be below the separatrix, while the latter measures the relative extent of the fluctuating energy compared to the initial value. We see that the areas of \(\xi_c^{-} < -0.1 \) or \(\Delta \xi < 2\) (dashed red lines) are occupied only by librating orbits, while the remaining area (coloured in light green) has all of the circulating orbits and only two librating orbits, which are the retrograde satellites Neso and Euporie that exhibit highly perturbed orbits.\(^2\) We will examine individual orbits below. Small values of \(\Delta \xi\) indicate that the fluctuation is much larger than \(E_c - E_s\); thus, it is harder to maintain a librating orbit, while \(-\xi_c^{-}\) measures the normalized distance from the separatrix, so larger values translate to stronger fluctuations.

\(^2\)The prograde satellite Carpo could sometimes be in this area, but is rarely realized in practice.
Finally, we stress that although $\epsilon_{SA}$ specifies the fluctuation strength (which provides $\epsilon_{c}^{\pm}$ via $\Delta J_{c} \propto \epsilon_{SA}$), correctly predicting whether a satellite will librate cannot be determined from $\epsilon_{SA}$ alone, and requires additional information on the relative distance from the separatrix (which provides $\epsilon_{c}$). Satellites with lower $\epsilon_{SA}$ are described well by the Brown Hamiltonian, but as we shall see, they could be misclassified (e.g. S2019_S01), while satellites with larger $\epsilon_{SA}$ are correctly classified (e.g. S2020_S05).

To summarize, we propose the following way to classify librating orbits:

1. From the initial conditions, calculate the energy of the total (secular and Brown) Hamiltonian in equation (11) and the dimensionless energy $\epsilon_{c}$ in equation (10). If $\epsilon_{c} < 0$, the orbit is not librating, otherwise
2. Calculate the fluctuating terms $E_{c}^{\pm}$ in equation (12) and $\epsilon_{c}^{\pm}$ in equation (15). If the minimum of $\epsilon_{c}^{\pm} > 0$ is positive, the orbit is librating, otherwise
3. Calculate the relative energy $\Delta \epsilon = \epsilon_{c}^{\pm} / (\epsilon_{c}^{-})$. If $\epsilon_{c}^{\pm} < -0.1$ or $\Delta \epsilon < 2$, the orbit is librating. Otherwise, it is most likely circulating, but direct N-body integration is required to confirm.

Generally, prograde satellites tend to be librating and follow the libration–circulation limits. Retrograde satellites, even though more numerous, tend to have highly perturbed orbits and even their librating orbits are unusual. We examine several individual satellites below.

5 ORBITS OF INDIVIDUAL SATELLITES

In this section, we closely examine the orbits of some ISs. In all the plots, the orbital elements are plotted in the variable plane (i.e. the orbital plane aligns with the Sun–planet orbit). In order to transform to the variable plane, we first obtain the orbital elements of the planet, $\omega_{p}$, $i_{p}$, and $\Omega_{p}$. Some satellites are close to the separatrix and their fate could be sensitive to the exact initial conditions. We have tested several satellite orbits with additional giant planets and
found no significant difference for the presented time-scales. Long-term integration of the full Solar system planets will lead to secular resonances between the planet and satellites’ longitude of pericentre precession (e.g. Saha & Tremaine 1993), so our numerical integration could deviate for much longer time-scales. Although our simplified three-body integrations agree with past detailed integrations of the Solar system, the ephemeris accuracy decreases over time (Emelyanov, Varfolomeev & Lainey 2022). Further observations are required for constant updates of the ephemeris.

After loading the planet and the satellites’ state vectors, we then use REBOUND’s built-in rotation function rebound.rotation.orbit() (https://rebound.readthedocs.io/en/latest/ipython_examples/rotations/), which uses quaternions. The Euler angles of the rotation matrix are directly specified to rotation_orbit(). Using the inverse rotation $P^{-1}_1(\phi_0) P^{-1}_2(\theta_0) P^{-1}_3(\omega_0)$ aligns the plane of the planetary orbit with the $XY$ reference plane (see e.g. section 2.8 of Murray & Dermott 1999). The inverse rotation is also applied on the state vector of the satellite before the integration.

In addition to the $N$-body integrations with REBOUND, we also run two instances of the secular code SECULAR (Grishin & Perets 2022), which solves the secular equations of motion and includes the extra terms from Luo et al. (2016), as the effective Brown Hamiltonian, for an arbitrary reference frame, as well as the octupole terms. External effects such as tidal dissipation and galactic tides are turned off. We compare direct $N$-body integrations with the secular Hamiltonian alone and also with the secular and Brown Hamiltonian. For the initial conditions, we use the $N$-body integration to generate averaged values of the orbital elements after one inner orbit.

In generating Figs 4–10, the energy in the bottom middle panels is calculated at each time-step $E_i(t)$. The fluctuating amplitudes $E_{ij}^\pm(t)$ (salmon lines) are averaged over the outer orbital period using PYTHON’S BOTTLENECK.MOVE package (https://bottleneck.readthedocs.io/en/latest/bottleneck.move.html). The phase portraits (bottom right panels) are similar to the phase portrait in Paper I.

5.1 Jovian satellites
Brozović & Jacobson (2017) studied the orbits of ISs and found that Euporie is librating. We confirm this result in Fig. 4 where the orbital elements are similar to their fig. 6, although we only have three-body integrations and not the full model of the Solar system.3 Euporie almost does not have any noticeable ZLK cycles, but remains to be librating. The eccentricity remains relatively low and the fluctuations in the energy are overpredicted (which causes the misclassification of Euporie). A satellite with higher eccentricity at the fixed point would have escaped to circulation, which makes

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3Also, our orbital elements are in the planet’s invariable plane while other studies usually present the orbital elements in the ecliptic reference frame, so small differences are expected.
its orbit unusual. The modulation in the energy suggests that other neglected terms may be important and/or the current expansion is inaccurate.

Fig. 5 shows the orbital evolution of Carpo and S2018_J4. Carpo seems to librate as expected, and the fluctuation in the energy follows the prediction; however, the Brown Hamiltonian model is still not accurate enough. S2018_J4 was discovered only recently and is not included in the Brozović & Jacobson (2017) analysis. Recently, it was reported that S2018_J4 has a ‘similar orbit’ to Carpo (Sheppard et al. 2023), but its dynamical evolution was not explicitly investigated. We suggest that S2018_J4 is also librating on an orbit very close to the boundary. $\omega_1$ seems to have a relatively large amplitude and the orbit spends a significant amount of time beyond the separatrix. Although both Carpo and S2018_J4 librate, their orbits are significantly different; S2018_J4 librates much slower, its argument of pericentre is librating around $3\pi/2$ and has a larger excess, and the minimal eccentricity attains much lower values.

To conclude, we see from Jovian satellites that the separatrix boundary in the Brown Hamiltonian is tighter for prograde orbits and correctly displays the actual phase portrait boundary. Even if the Brown Hamiltonian model is not accurate, it is still a much better approximation than the secular Hamiltonian. This is also striking for the retrograde satellite Euporie, where the secular Hamiltonian does not have a fixed point or librating orbits at all, while in the Brown Hamiltonian the orbit is at least contained to the separatrix.
5.2 Saturnian satellites

Saturn has the largest number of ISs. Jacobson et al. (2022) found that three of them are in the ZLK librating mode. We confirm that Kiviuq is in librating mode (top panel of Fig. 6), and its evolution is similar to fig. 2 of Jacobson et al. (2022). The same is true for Ijiraq (not shown) and S2004_S31, where its orbital elements evolve similarly to fig. 4 of Jacobson et al. (2022). Its orbit is qualitatively similar to
Figure 7. Same as Fig. 5 but for different Saturn’s satellites.

S2018_J4 with a large amplitude variation of $\omega_1$ and extending well beyond the separatrix. The correspondence is worse, mainly because the initial energy estimated from the Brown Hamiltonian $\langle E_B \rangle$ is further from the separatrix than in reality.

Sheppard et al. (2023) suspected that S2005_S04 could be part of the Inuit group, but it has a significantly higher inclination. Indeed, the instantaneous inclination to the ecliptic reference frame is 52.6 deg, ~3 deg larger than the rest of the Inuit group. However, when we rotate to the invariable plane, the orbit of S2005_S04 is very similar to Ijiraq, and both satellites are in ZLK libration mode. Because the longitude of ascending nodes $\Delta \Omega$, of the two satellites is quite different, it translated to a noticeable difference in the ecliptic reference frame.

Besides the known librating satellites and the suggestion that S2005_S04 is also an Inuit-group librating satellite, we also propose that S2020_S01 and S2020_S04 are librating (Fig. 7). S2020_S01 is part of the Inuit group and shared similar orbital elements with Kiviuq and Ijiraq (Sheppard et al. 2023), but it was not explicitly mentioned that it is librating.

In addition, Saturn has a set of satellites that are very close to the boundary, but are circulating. Two of them (S2019_S01 and Tarqeq) appear in Fig. 1 as suspected, while Siarnaq’s current initial condition is just outside the separatrix and is not classified as suspected. We see in Fig. 8 that although the Brown Hamiltonian secular code (green lines) captures the long-term evolution better, the classification is inaccurate since both secular codes predict a librating orbit. Nevertheless, the boundary of the separatrix from the Brown Hamiltonian fits much better to the orbital evolution.

5.3 Uranian satellites

The only prograde IS of Uranus is Margaret, which is also the only one librating (Brozovic & Jacobson 2009; Brozović & Jacobson...
Figure 8. Same as Fig. 5 but for circulating Saturn’s satellites.

The top panel of Fig. 9 depicts Margaret’s orbital evolution and is consistent with fig. 24 of Brozović & Jacobson (2009) and fig. 5 of Brozović & Jacobson (2022). Margaret by far has the lowest $j_1$ value and can reach high eccentricities up to $\sim 0.9$. Margaret has the strongest resonance of all the prograde satellites in the sense that it is the furthest from the separatrix in dimensionless $\mathcal{E}_c$ energy units.
Paper I found a potential hypothetical family of highly inclined, highly eccentric, librating orbits that can extend far beyond the standard stability limit. This was also speculated by Carruba et al. (2002) on similar grounds. Margaret’s $\epsilon_{SA}$ is relatively small for such a population, but it can still be a representative member of such a population due to its highly eccentric and stable orbit.

5.4 Neptunian satellites

Neptune has two known librating satellites, Sao and Neso (Brozović et al. 2011). While Sao is prograde and has a relatively standard orbit, similar to Saturn’s Inuit group, Neso is located further out and has a retrograde highly perturbed orbit. Our integrations are shown in Fig. 10 and are consistent with fig. 5 of Brozović & Jacobson (2022). Note that the ZLK Hamiltonian misclassified Neso’s orbit and the Brown Hamiltonian is required to correctly recover its librating orbit, even though neither of the secular models is accurate enough.

The outermost satellite is S2021_N1, discovered only recently. We show that it is also in ZLK libration (bottom panel of Fig. 10), but the orbit slightly differs from Neso. While Neso avoids reaching high energies and its phase space is ‘doughnut shaped’, S2021_N1 easily covers the phase space’s fixed point. On the other hand, its energy fluctuations are much smaller, and it crosses the separatrix only for short periods in the highly eccentric phases. The argument of pericentre also has a smaller libration amplitude compared to Neso, which is attributed to lower $j_z$ values. Using the ZLK Hamiltonian completely misses S2021_N1’s phase portrait, and the addition of the Brown Hamiltonian makes it contained within the separatrix.

The official announcement was made on 2024 February 23: https://minorplanetcenter.net/mpec/K24/K24DB2.html.

The Wikipedia page for S2021_N1 states that the argument of pericentre librates around 90 deg with a reference to Brozović & Jacobson (2022), but S2021_N1 is not mentioned or included in their analysis.
Figure 10. Same as Fig. 5 but for different Neptune's satellites.
The bottom panel of Fig. 9 shows the orbital evolution of the circulating satellite Psamathe, which has a highly perturbed and unusual trajectory in phase space. Neither of the secular models can describe Psamathe’s orbital evolution.

6 SUMMARY AND CONCLUSIONS

We studied analytically and numerically the long-term evolution of triple systems of mild hierarchy, where the period ratio between the inner and outer orbits is not too small. Our conclusions are summarized below:

(i) We find that all of the librating satellites except Euporie are within the zone allowed for libration (Fig. 1). Most of the librating satellites are prograde and they can be very close to the librating–circulating boundary (equation 6).

(ii) We formulate a criterion that predicts whether a satellite’s argument of pericentre will be librating based on its initial conditions (Fig. 2). The criterion utilizes the effective energy conservation given by the secular and Brown Hamiltonians (equation 11) and the magnitude of the osculating energy (equation 12).

(iii) While the prograde satellites mostly follow the simplified criterion (Section 4.1), the highly perturbed ISs require a more refined criterion (Section 4.2). The exact process is described at the end of Section 4. Using this criterion, we were able to confirm that all previously known ISs are librating. In addition, we find that S2018_J4, S2005_S04, S2020_S01, S2020_S04, and S2021_N1 are also librating. Future observations and missions would increase our sample and test our predictions.

(iv) Although retrograde orbits are more numerous and have more available parameter space, the vast majority of librating satellites are prograde. The extra energy from the Brown Hamiltonian (and its osculating boundaries) correctly predicts the satellite’s motion in the $e_1-\psi_1$ phase space (Figs 4–10), except Euporie, whose osculating magnitude is greatly overestimated.

(v) Even if the evolution is inaccurate, the librating satellites are contained within the libration zones (right panels of Figs 4–10), and the circulating satellites are out of the libration zones (right panels of Fig. 8).

Future work will be able to better model the retrograde satellites where the Brown Hamiltonian is inaccurate. One option is to construct a simplified version of a higher order theory (Lei 2019) where the first averaging is done after the canonical transformation. Simplified models will not require literal expansions and will converge to any eccentricity. Future data on ISs and binary minor planets in the asteroid and Kuiper belt will put our theory under examination. Additional perturbations from other bodies or tidal and rotational forces could potentially change our results.

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DATA AVAILABILITY

The data used in this paper are public and the methods are publicly available in the GitHub repository (https://github.com/eugeneg88/fixed_points_brown).

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